A GENERALIZATION OF STEINHAGEN'S THEOREM

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Abstract. The theorem of Steinhagen establishes a relation between inradius and width of a convex set. The half of the width can be interpreted as the minimum of inradii of all 1-dimensional orthogonal projections of a convex set. By considering i-dimensional projections we obtain series of i-dimensional inradii. In this paper we study some relations between these inradii and by this we find a natural generalization of Steinhagen's theorem.

Further we show in the course of our investigation that the minimal error of the triangle inequality for a set of vectors cannot be too large.

1. Introduction and basic notation

Throughout this paper E^d denotes the d-dimensional Euclidean space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, and the set of all convex sets $K\subset E^d$ — not necessarily bounded — is denoted by $\mathcal{K}^d$. The affine (convex, linear) hull of a subset $P\subset E^d$ is denoted by $\mathrm{aff}(P)$ ($\mathrm{conv}(P)$, $\mathrm{lin}(P)$), and $\dim(P)$ denotes the dimension of the affine hull of P. The interior of P with respect to the affine hull of P is denoted by $\mathrm{relint}(P)$.

The set of all i-dimensional linear subspaces of E^d is denoted by $\mathcal{L}_i^d$. $L^\perp$ denotes for $L\in\mathcal{L}_i^d$ the orthogonal complement and for $K\in\mathcal{K}^d$, $L\in\mathcal{L}_i^d$ the orthogonal projection of K onto L is denoted by $K|L$.

For an affine space $A\subset E^d$ and a convex set $K\subset A$ we denote by $r(K;A)$ the inradius of K with respect to the Euclidean space A, and $\Delta(K)$ denotes the width of K, i.e. the minimal distance of two supporting hyperplanes of K. Clearly $r(K)=r(K;E^d)$ is the usual inradius, i.e. the maximal radius of a d-dimensional ball contained in K. For a detailed description of these functionals we refer to [BF]. With the above notation we can define the following i-dimensional inradii

Definition 1.1. *For $K\in\mathcal{K}^d$ and $1\le i\le d$ let*

$$r_i(K)=\min\{r(K|L;L)\mid L\in\mathcal{L}_i^d\}.$$

We obviously have $r_{i+1}(K)\le r_i(K)$ and $r_d(K)=r(K)$, $r_1(K)=\Delta(K)/2$. The classical theorem of Steinhagen [St] states a relation between the inradius and the width of a convex set. In our notation his result reads as follows

1991 *Mathematics Subject Classification.* 52A40, 52A20.

Key words and phrases. inradius, width.

Steinhagen's Theorem. *Let $K \in \mathcal{K}^d$ with $r_1(K) < \infty$. Then*

$$r_d(K)/r_1(K) \geq \begin{cases} \sqrt{d+2}/(d+1), & \text{if } d \text{ even}, \\ 1/\sqrt{d}, & \text{if } d \text{ odd}. \end{cases} \tag{1.1}$$

We remark that there are several proofs of Steinhagen's theorem, see e.g. [G], [J], [E, pp. 112]. Here we study more generally the relations between r_i and r_j. We start with the investigation of the inradii of simplices $T \in \mathcal{K}^d$, because the general case can be reduced to this problem. First, in section 2, we identify a finite subset of i-dimensional planes $\mathcal{L}_{\mathcal{C}_i^d} \subset \mathcal{L}_i^d$ which yields the minimal inradius, i.e. $r_i(T) = \min\{r(T|L; L) \mid L \in \mathcal{L}_{\mathcal{C}_i^d}\}$. In the third section we describe a formula for the ratio $r(T)/r_i(T)$ in terms of some sizes of the simplex T. Unfortunately we can only use this formula to determine the extreme simplices in the cases $i = d-1$ and $i = 1$. For the remaining cases our results are almost certainly not best possible. In the last section we transfer our results to arbitrary convex bodies and to arbitrary ratios $r_i(K)/r_j(K)$. By this we obtain a new proof and a generalization of Steinhagen's result (cf. Theorem 5.1).

An auxiliary result in our investigation which is of some interest in its own is a certain converse of the triangle inequality:
For a set of vectors $\{a^1, \ldots, a^n\}$ and an integer $i < n$ there is a subset $\{a^{j_1}, \ldots, a^{j_i}\}$ such that the error $\sum_{k=1}^{i} \|a^{j_k}\| - \|\sum_{k=1}^{i} a^{j_k}\|$ is less than a constant depending on n, i and $\sum_{i=1}^{n} \|a^i\|$, and for a suitable subset $\{a^{l_1}, \ldots, a^{l_i}\}$ the length of $\|\sum_{k=1}^{i} a^{l_k}\|$ is not too small.

Let us remark that generalizations on inradius and circumradius offer new aspects of known and some completely new opportunities in convexity. A first systematic study is in Henk [H1], where e.g. the corresponding generalization of Jung's theorem is proved (see also [H2]) and a more comprehensive bibliography is given. At this point we only want to mention a quite recent result by Ball [B] who showed among other things that the maximal i-dimensional ball contained in a regular simplex lies in an i-face of the regular simplex.

To start our investigations we need some more notation. In this paper $T \subset E^d$ denotes a d-dimensional simplex with vertices $x^0, \ldots, x^d$. The facets of T are denoted by $F^i = \mathrm{conv}(\{x^0, \ldots, x^{i-1}, x^{i+1}, \ldots, x^d\})$, $0 \leq i \leq d$. The outward normal vector of F^i weighted by the volume of F^i is denoted by u^i. It is well known that [BF, pp. 118]

$$\sum_{i=0}^{d} u^i = 0, \tag{1.2}$$

$$r(T) \cdot \sum_{i=0}^{d} \|u^i\| = d \cdot V(T). \tag{1.3}$$

For an integer d let $M_d = \{0, 1, \ldots, d\}$ and $\mathcal{M}_i^d$ be the set of all subsets of M_d with cardinality i, $1 \leq i \leq d$. We say that $C = \{C_0, \ldots, C_i\}$ is an *i-partition* of the set $M_d = \{0, 1, \ldots, d\}$ iff $C_0, \ldots, C_i$ are non empty subsets of M_d and every element of M_d belongs to one and only one C_j, $j \in \{0, \ldots, i\}$, i.e.

$$C = \{C_0, \ldots, C_i\} \text{ is an } i\text{-partition} :\Longleftrightarrow \cup_{j=0}^{i} C_i = M_d \wedge C_k \cap C_l = \emptyset,\ 0 \leq k < l \leq i.$$

Let $\mathcal{C}_i^d$ be the sets of all i-partitions with respect to M_d, $0 \leq i \leq d$, and for a subset $S \subset M_d$ let L_S be the $(d + 1 - |S|)$-dimensional linear space which is the orthogonal complement of the affine hull of the face $\mathrm{conv}(\{x^j \mid j \in S\})$ of T. For $C = (C_0, \ldots, C_i) \in \mathcal{C}_i^d$ let $L_C = \cap_{j=0}^{i} L_{C_j}$, $\mathcal{L}_{\mathcal{C}_i^d} = \{L_C \mid C \in \mathcal{C}_i^d\}$ and $T^C = T|L_C$ be the orthogonal projection of T onto L_C.

By $V(P)$ for $P \subset E^d$ we understand the volume of P with respect to $\mathrm{aff}(P)$. Further for a real number x we denote by $\lceil x \rceil$ ($\lfloor x \rfloor$) the smallest (largest) integer $\geq$ ($\leq$) x. For abbreviation we set for $1 \leq i \leq d$

$$\alpha(d,i) = \frac{i}{d+1} + \sqrt{\frac{(d+1-i)i}{(d+1)^2 d}},$$

$$\beta(d,i) = m \cdot \sqrt{\frac{\lceil l \rceil (d+1-\lceil l \rceil)}{(d+1)^2 d}} + (i+1-m) \cdot \sqrt{\frac{\lfloor l \rfloor (d+1-\lfloor l \rfloor)}{(d+1)^2 d}},$$

with $l = (d+1)/(i+1)$ and $d+1 \equiv m \mod (i+1)$, $0 \leq m \leq i$.

We have $\alpha(d,d-1) = \beta(d,d-1)$ and $\beta(d,1) = (d+2)^{1/2}/(d+1)$ if d even, $\beta(d,1) = (d)^{-1/2}$ else, cf. (1.1).

Finally, for $n \in \mathbb{N}$ and $i \in \mathbb{Z}$ let $b_i^n = \binom{n}{i}$ if $i \geq 0$, $b_i^n = 0$ else. æ

2. Minimal inradii of simplices

By the definition of $C = (C_0, \ldots, C_i) \in \mathcal{C}_i^d$ we see that L_C is an i-dimensional plane and T^C is a simplex with the property that $x^j|L_C$ is a vertex of T^C for all vertices of T. Moreover the set $\mathcal{L}_{\mathcal{C}_i^d}$ is the finite set of all i-dimensional spaces $L \in \mathcal{L}_i^d$ with the property that $T|L$ is an i-simplex and the images of all vertices of T under the orthogonal projection of T onto L are vertices of $T|L$.

Theorem 2.1. shows that we only have to consider the projections onto elements of $\mathcal{L}_{\mathcal{C}_i^d}$ to find $r_i(T)$. Before we prove this we study the following special case

Lemma 2.1. *Let $L \in \mathcal{L}_{d-1}^d \backslash L_{\mathcal{C}_{d-1}^d}$ and $T|L$ be a $(d-1)$-simplex. Then*

$$r_{d-1}(T) < r(T|L; L).$$

Proof. Let $L = \{x \in E^d \mid \langle u, x \rangle = 0\}$ with $\|u\| = 1$ and $y^i = x^i|L$, $0 \leq i \leq d$. By our assumption we may assume $y^d \in \mathrm{relint}(\mathrm{conv}(\{y^0, \ldots, y^i\}))$ for some $i \geq 1$. Now, let $v^1 = x^1 - x^0/\|x^1 - x^0\|$ and $v^2 = \lambda u + \mu v^1$ for some $\lambda, \mu \in \mathbb{R}$ such that $\langle v^1, v^2 \rangle = 0$ and $\|v^2\| = 1$. In the following we study the projection of T onto the set of hyperplanes $L(\phi) = \{x \in E^d \mid \langle v(\phi), x \rangle = 0\}$, $\phi \in (-\pi/2, \pi/2)$, with $v(\phi) = \cos(\phi) v^2 - \sin(\phi) v^1$.

By construction we have $L(\overline{\phi}) = L$ for a suitable $\overline{\phi}$, and there exists an $\epsilon > 0$ such that $T|L(\phi) = F^d|L(\phi)$ for all $\phi \in (\overline{\phi} - \epsilon, \overline{\phi} + \epsilon)$.

It remains to show that $r(\phi) = r(F^d|L(\phi); L(\phi))$ has no minimum at $\overline{\phi}$. To this end let $f^i = \mathrm{conv}(\{x^0, \ldots, x^{i-1}, x^{i+1}, \ldots, x^{d-1}\})$, $0 \leq i \leq d-1$, denote the facets of F^d. By (1.3) we obtain

$$\text{(2.1)} \qquad \frac{1}{r(\phi)} = \frac{1}{d-1} \frac{\sum_{i=0}^{d-1} V(f^i|L(\phi))}{V(F^d|L(\phi))} = \sum\nolimits_{i=0}^{d-1} \frac{1}{h_i(\phi)},$$

where $h_i(\phi)$ denotes the distance between $x^i|L(\phi)$ and $\mathrm{aff}(f^i|L(\phi))$. To evaluate this expression let f be an arbitrary facet of F^d and $L_f \in \mathcal{L}_{d-2}^d$ be the linear subspace associated with $\mathrm{aff}(f)$. Now we determine the intersection of $L_f^\perp$ with $L(\phi)$. To this end let u^1, u^2 be an orthonormal basis of $L_f^\perp$ such that u^1 is orthogonal to F^d. Then for some $\psi(\phi) \in [0,\pi]$ we have $\sin(\psi(\phi))u^1 + \cos(\psi(\phi))u^2 \in L(\phi)$. This means

$$\sin(\psi(\phi)) \cdot \langle \cos(\phi)v^2, u^1\rangle + \cos(\psi(\phi)) \cdot \big(\langle \cos(\phi)v^2, u^2\rangle - \langle \sin(\phi)v^1, u^2\rangle\big) = 0.$$

Solving for $\cos^2(\psi(\phi))$ gives

$$\cos^2(\psi(\phi)) = \frac{\cos^2(\phi)\langle v^2, u^1\rangle^2}{\cos^2(\phi)\langle v^2, u^1\rangle^2 + (\cos(\phi)\langle v^2, u^2\rangle - \sin(\phi)\langle v^1, u^2\rangle)^2}. \tag{2.2}$$

If x, y denote vertices of F^d with $x \in f$ and $y \notin f$, it follows for the distance $h_f(\phi)$ between $y|L(\phi)$ and $\mathrm{aff}(f|L(\phi))$ that

$$h_f(\phi)^2 = \langle \sin(\psi(\phi))u^1 + \cos(\psi(\phi))u^2, x - y\rangle^2 = \cos^2(\psi(\phi)) \cdot \langle u^2, x - y\rangle^2.$$

Together with (2.1) and (2.2) we obtain immediately

$$\frac{1}{r(\phi)} = \sum_{j=0}^{d-1} \frac{(\lambda_1^j \cos^2(\phi) + \lambda_2^j \sin(\phi)\cos(\phi) + \lambda_3^j \sin^2(\phi))^{1/2}}{|\cos(\phi)|},$$

where $\lambda_i^j \in \mathbb{R}$, $1 \le i \le 3$, $0 \le j \le d-1$, denote constants independent of ϕ. Next we consider the map $(-\pi/2, \pi/2) \to (-\infty, \infty)$ with $\sin(\phi) = x/(x^2+1)^{1/2}$ and $\cos(\phi) = 1/(x^2+1)^{1/2}$. This transformation yields

$$\frac{1}{r(x)} = \sum_{i=0}^{d-1} \sqrt{q_i(x)},$$

for certain quadratic functions $q_i(x)$. From their geometrical definitions we have $q_i(x) > 0$ for all $x \in (-\infty, \infty)$. Thus $\sqrt{q_i(x)}$ are strictly convex functions and hence $1/r(x)$. This means that $r(\phi)$ cannot attain a minimum in the interior of its domain. This shows the assertion. □

Now we are able to prove the general case

Theorem 2.1. *Let $T \subset E^d$ be a d-dimensional simplex. Then for $1 \le i \le d-1$*

$$r_i(T) = \min\{r(T|L; L) \mid L \in \mathcal{L}_{\mathcal{C}_i^d}\}.$$

Proof. Let $L \in \mathcal{L}_i^d \backslash \mathcal{L}_{\mathcal{C}_i^d}$ and let $y^i = x^i|L$, $0 \le i \le d$. We shall prove $r(T|L;L) > r_i(T)$.

To do this we have to distinguish the two cases that $T|L$ is an i-simplex or not. In the first case we will find by a suitable projection of T an $i+1$-simplex $\tilde{T}$ and an i-dimensional plane $\tilde{L}$ with $T|L = \tilde{T}|\tilde{L}$ such that $\tilde{L}$ satisfies the assumption of Lemma 2.1. with $d = i+1$. If $T|L$ is not a simplex we construct by the information of a certain circumscribed simplex $\overline{T}$ of $T|L$ a simplex $\tilde{T}$ with $T \subset \tilde{T}$ and $r(T|L;L) = r(\tilde{T}|L;L)$. Then we show that $r_i(\tilde{T}) < r(T|L;L)$.

First, suppose $\overline{T} = T|L$ is an i-simplex with vertices $y^0, \dots, y^i$. Let $A_j = \{k \in M_d \mid x^k|L = y^j\}$, $0 \le j \le i$, and $A = M_d \setminus \cup_{j=0}^{i} A_j$. By our assumption we may assume $A = \{d-p, \dots, d\}$ for a suitable $p \ge 0$. For every x^j, $j \in A$, there exists a unique $z^j \in \operatorname{conv}(\{x^0, \dots, x^i\})$ with $z^j|L = x^j|L$. Let $u^j = x^j - z^j$ and let L_j be the hyperplane with normal vector u^j, $j \in A$. We have

$$L = \left(\cap_{j=0}^{i} L_{A_j}\right) \cap \left(\cap_{j \in A} L_j\right).$$

Hence $\overline{T} = (T|\overline{L})|(L_d \cap \overline{L})$ with $\overline{L} = (\cap_{j=0}^{i} L_{A_j}) \cap (\cap_{j \in A \setminus \{d\}} L_j)$. $\tilde{T} = T|\overline{L}$ is an $(i{+}1)$-simplex with vertices $x^j|\overline{L}$, $j = 0, \dots, i$, and $x^d|\overline{L}$. By the previous Lemma we know that there is an i-dimensional plane $H \subset \overline{L}$ with $r(T|H; H) = r(\tilde{T}|H; H) < r(\tilde{T}|(L_d \cap \overline{L}); L) = r(T|L; L)$.

Now, suppose that $P = T|L$ is not a simplex and let z be the center of the insphere of P. Then there exist points $z^0, \dots, z^k$ in the intersection of the boundary of P with the insphere, such that $z \in \operatorname{relint}(\operatorname{conv}(\{z^0, \dots, z^k\}))$ [BF]. We choose a set of minimal cardinality and from CARATHEODORY'S theorem we have $k \le i$.

Let $\overline{L} = \operatorname{lin}(\{z^1 - z^0, \dots, z^k - z^0\})$ and let $\overline{y}^j = x^j|\overline{L}$. The insphere of $\overline{P} = T|\overline{L}$ is the projection of the insphere of $T|L$ onto $\overline{L}$. Further the affine hull of the facets $\overline{f}^j$ of $\overline{P}$ with $z^j|\overline{L} \in \overline{f}^j$ define a circumscribed simplex $\overline{T}$ of $\overline{P}$ with

$$r(\overline{T}; \overline{L}) = r(\overline{P}; \overline{L}) = r(T|L; L).$$

Now we construct from $\overline{T}$ a new simplex $\tilde{T}$ in the following way:

For $p, q \in M_d$ we consider the ray $[\overline{y}^p, \overline{y}^q)$ emanating from $\overline{y}^p$. Let $\overline{g}$ denote the face of $\overline{T}$ with minimal dimension containing $\overline{y}^p, \overline{y}^q$. Then let $\tilde{y}^q$ be the point of intersection of the boundary of $\overline{g}$ with $[\overline{y}^p, \overline{y}^q)$. There is an unique $\tilde{x}^q \in [x^p, x^q)$ with $\tilde{x}^q|\overline{L} = \tilde{y}^q$. For $\tilde{T} = \operatorname{conv}(\{x^0, \dots, x^{q-1}, \tilde{x}^q, x^{q+1}, \dots, x^d\})$ we have $T \subset \tilde{T}$ and

$$r(\tilde{T}|\overline{L}; \overline{L}) = r(T|\overline{L}; \overline{L}).$$

After a finite number of such steps we get a simplex $\tilde{T}$ with $\tilde{T}|\overline{L} = \overline{T}$ and for every vertex $\tilde{x}^q$ of $\tilde{T}$ we have $\tilde{x}^q|\overline{L}$ is a vertex of $\overline{T}$.

In the case $k = i$ we stop the process before executing the last step. Then $\tilde{T}|L$ is a simplex, but there is still a vertex $\tilde{x}$ of $\tilde{T}$, such that $\tilde{x}|\overline{L}$ is not a vertex of $\overline{T}$. On account of the first case it follows

$$r_i(T) \le r_i(\tilde{T}) < r(\tilde{T}|\overline{L}; \overline{L}) = r(T|\overline{L}; \overline{L}) = r(T|L; L),$$

and this is a contradiction.

In the case $k < i$ we carry the process to its end. Then every $\tilde{y}^q = \tilde{x}^q|\overline{L}$ is a vertex of $\overline{T}$. We may assume $\overline{T} = \operatorname{conv}(\{\tilde{y}^0, \dots, \tilde{y}^k\})$. Further we denote for $j = k{+}1, \dots, d$ by $p_j \in \{0, \dots, k\}$ the index with $\tilde{y}^j = \tilde{y}^{p_j}$. Now we choose the i-dimensional plane L', which is orthogonal to the edges $\operatorname{conv}(\{\tilde{x}^j, \tilde{x}^{p_j}\})$, $j = i+1, \dots, d$. Now, $\tilde{T}|L'$ is a simplex and $(\tilde{T}|L')|\overline{L} = \overline{T}$. As the inradius of a simplex is always strictly less than the inradius of its projection we obtain

$$r(T|L; L) = r(\tilde{T}|\overline{L}; \overline{L}) = r((\tilde{T}|L')|\overline{L}; \overline{L}) > r(\tilde{T}|L'; L') \ge r(T|L'; L').$$

Thus $r(T|L; L)$ is not minimal. □

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3. A formula for the minimal inradii of simplices

Let $C = (C_0, \dots, C_i) \in \mathcal{C}_i^d$. By the definition of L_C we have that $T_C = T|L_C$ is a i-dimensional simplex with vertices $x^{C_0}, \dots, x^{C_i}$ where $x^{C_j} = x^k|L_C$ for all $k \in C_j$, $0 \le j \le i$. Again, let $F^{C_j} = \mathrm{conv}(\{x^{C_0}, \dots, x^{C_{j-1}}, x^{C_{j+1}}, \dots, x^{C_i}\})$ denote the facets of T_C and u^{C_j} the outward normal vector of F^{C_j} weighted by the volume of F^{C_j}, $0 \le j \le i$.

Lemma 3.1. *Let $C = (C_0, \dots, C_i) \in \mathcal{C}_i^d$. With the notation above we have for $0 \le j \le i$*

$$u^{C_j} = \left(\frac{iV(T_C)}{dV(T)}\right) \cdot \sum_{k \in C_j} u^k.$$

Proof. The proof will be done by induction with respect to i. Clearly, the relations hold for $i = d$. So let $i < d$. Then we may assume that the set C_i has at least two elements and hence let $\overline{C}_i, \overline{C}_{i+1}$ be two non empty subsets of C_i with $C_i = \overline{C}_i \cup \overline{C}_{i+1}$ and $\overline{C}_i \cap \overline{C}_{i+1} = \emptyset$. Now, let $\overline{C}_j = C_j$ for $0 \le j \le i-1$ and $\overline{C} = (\overline{C}_0, \dots, \overline{C}_{i+1}) \in \mathcal{C}_{i+1}^d$. We have $T_C = T_{\overline{C}}|L$, where $L \subset L_{\overline{C}}$ is the i-dimensional plane with normal vector $x^{\overline{C}_i} - x^{\overline{C}_{i+1}}$. By elementary linear algebra we get with $\gamma = \|x^{\overline{C}_i} - x^{\overline{C}_{i+1}}\|$

$$\begin{aligned} V(T_{\overline{C}}) &= (\gamma/(i+1)) \cdot V(T_C) \\ V(F^{\overline{C}_j}) &= (\gamma/i) \cdot V(F^{C_j}), \quad 0 \le j \le i-1. \end{aligned} \tag{3.1}$$

Now, the outward normal vectors of the facets of $T_{\overline{C}}$ containing $x^{\overline{C}_i}$ and $x^{\overline{C}_{i+1}}$ are parallel to the plane L and thus for $0 \le j \le i-1$

$$u^{\overline{C}_j} = (\gamma/i) \cdot u^{C_j}.$$

By (1.2) we find $u^{C_i} \cdot (\gamma/i) = u^{\overline{C}_i} + u^{\overline{C}_{i+1}}$. On account of the induction hypothesis for $\overline{C}$ we get with (3.1) the desired relations for $C \in \mathcal{C}_i^d$. □

By this Lemma and Theorem 2.1. we get the following formula

Theorem 3.1. *Let T be a d-dimensional simplex. Then for $1 \le i \le d-1$*

$$\frac{r(T)}{r_i(T)} = \max_{C=(C_0,\dots,C_i)\in\mathcal{C}_i^d} \frac{\sum_{j=0}^i \|\sum_{k\in C_j} u^k\|}{\sum_{k=0}^d \|u^k\|}.$$

Proof. On account of Theorem 2.1. we have $r_i(T) = \min_{C\in\mathcal{C}_i^d} r(T_C; L_C)$. Hence by Lemma 3.1. and (1.3) we have for $C = (C_0, \dots, C_i) \in \mathcal{C}_i^d$

$$r(T)/r(T_C; L_C) = \sum\nolimits_{j=0}^i \|\sum\nolimits_{k\in C_j} u^k\| / (\sum\nolimits_{k=0}^d \|u^k\|).$$

□

In the next section we use this formula to give estimates of $r(T)/r_i(T)$ for arbitrary simplices. But in particular for a regular simplex we get

Lemma 3.2. *Let $S \subset E^d$ be a d-dimensional regular simplex. Then for $1 \le i \le d$*

$$r(S)/r_i(S) = \beta(d,i).$$

Proof. We may assume that the surface area of S is normed to 1. Hence we have $\|u^k\| = 1/(d+1)$, $0 \le k \le d$, and by (1.2) $\langle u^k, u^j\rangle = -1/(d(d+1)^2)$, $0 \le k < j \le d$. Thus for $C = (C_0, \dots, C_i) \in \mathcal{C}_i^d$ we find by Theorem 3.1.

$$\frac{r(S)}{r(S_C; L_C)} = \sum\nolimits_{j=0}^{i} \sqrt{\frac{|C_j|(d+1-|C_j|)}{(d+1)^2 d}}. \tag{3.2}$$

Now, suppose that $|C_0| > |C_1| + 1$ and let $u \in C_0$. We consider the new i-partition $\overline{C} = (\overline{C}_0, \dots, \overline{C}_i)$ with $\overline{C}_l = C_l$, $2 \le l \le i$, $\overline{C}_0 = C_0 \backslash \{u\}$ and $\overline{C}_1 = C_1 \cup \{u\}$ and show $r(T_{\overline{C}}; L_{\overline{C}}) < r(T_C; L_C)$. If we set $|C_0| = |C_1| + x$ and $g(x) = \sqrt{(|C_1| + x)(d + 1 - |C_1| - x)}$, then on account of (3.2) we have to prove for $x > 1$

$$g(x) + g(0) < g(x-1) + g(1).$$

Clearly, for $x = 1$ we have equality. So it suffices to prove $g'(x) < g'(x-1)$. This is an immediate consequence of $g''(x) < 0$ which is easily checked. By Theorem 3.1. this means that for the minimal i-inradius we only have to consider i-partitions $C = (C_0, \dots, C_i)$ with the property $|\,|C_k| - |C_j|\,| \le 1$ for $0 \le k < j \le i$. But such a partition is uniquely determined with respect to the cardinality of the sets C_i. We have m sets with cardinality $\lceil l \rceil$ and $(i + 1 - m)$ sets with cardinality $\lfloor l \rfloor$. This shows the Lemma. □

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4. Estimating the minimal inradii of simplices

To use the formula of Theorem 3.1 we will study two more general relations, which may be considered as converses to the well known triangle inequality.

Theorem 4.1. *Let $a^0, \dots, a^n \in E^d$ with $\sum_{j=0}^n \|a^j\| > 0$. Then we have for $i = 1, \dots, n$*

$$\text{i)} \quad \min_{M_i \in \mathcal{M}_i^n} \left(\sum_{j \in M_i} \|a^j\| - \|\sum_{j \in M_i} a^j\| \right) \le \left(\frac{i}{n+1} - \sqrt{\frac{i(n+1-i)}{(n+1)^2 n}} \right) \cdot \sum_{j=0}^n \|a^j\|,$$

$$\text{ii)} \quad \max_{M_i \in \mathcal{M}_i^n} \|\sum_{j \in M_i} a^j\| \ge \sqrt{\frac{i(n+1-i)}{(n+1)^2 n}} \cdot \sum_{j=0}^n \|a^j\|,$$

and if equality holds in i) or ii) for some $i \in \{2, \dots, n\}$, then $\sum_{j=0}^n a^j = 0$ and $\|a^j\| = \sum_{j=0}^n \|a^j\|/(n+1)$ for $0 \le j \le n$.

Proof. For simplification we may assume $\sum_{j=0}^n \|a^j\| = 1$ and denote by $c_{n,i}$ $(d_{n,i})$ the right hand side of the inequality i) (ii)).

Now suppose $\|\sum_{j\in M_i} a^j\| < \sum_{j\in M_i}\|a^j\| - c_{n,i}$ for all $M_i\in\mathcal{M}_i^n$. By taking the square and subtracting $\sum_{j\in M_i}\|a^j\|^2$ we get

$$2\sum_{k,j\in M_i,\,k\neq j}\langle a^k,a^j\rangle < c_{n,i}^2 - 2c_{n,i}\sum_{j\in M_i}\|a^j\| + 2\sum_{k,j\in M_i,\,k\neq j}\|a^k\|\cdot\|a^j\| \tag{4.1}$$

Summing over all $M_i\in\mathcal{M}_i^n$, adding on both sides b_{i-2}^{n-1} times $\sum_{j=0}^n\|a^j\|^2$ and dividing by b_i^{n+1} yields with respect to $\sum_{j=0}^n\|a^j\| = 1$

$$\frac{i(i-1)}{n(n+1)}\cdot\|\sum_{j=0}^n a^j\|^2 < c_{n,i}^2 - 2c_{n,i}\frac{i}{n+1} + \frac{i(i-1)}{n(n+1)}. \tag{4.2}$$

By the definition of $c_{n,i}$ the right hand side is equal to 0 and this is a contradiction.

If there is equality in i) we must have $\sum_{j\in M_i}\|a^j\| - \|\sum_{j\in M_i} a^j\| = c_{n,i}$ for all $M_i\in\mathcal{M}_i^n$ and by (4.2) $\sum_{j=0}^n a^j = 0$. In particular we have equality in (4.1). Now summing (4.1) over all $M_i\in\mathcal{M}_i^n$ containing a fixed number l (say $l=0$) yields

$$\begin{aligned}
&b_{i-2}^{n-1}\cdot 2\sum\nolimits_{j=1}^n\langle a^0,a^j\rangle + b_{i-3}^{n-2}\cdot 2\sum\nolimits_{1\le k<j\le n}\langle a^k,a^j\rangle =\\
&b_{i-1}^n\cdot c_{n,i}^2 - 2c_{n,i}\left(b_{i-1}^n\cdot\|a^0\| + b_{i-2}^{n-1}\cdot\sum\nolimits_{j=1}^n\|a^j\|\right) +\\
&b_{i-2}^{n-1}\cdot 2\sum\nolimits_{j=1}^n\|a^0\|\cdot\|a^j\| + b_{i-3}^{n-2}\cdot 2\sum\nolimits_{1\le k<j\le n}\|a^k\|\cdot\|a^j\|.
\end{aligned}$$

Using the identities $(\sum_{j=0}^n\|a^j\|)^2 = 1$ and $\langle\sum_{j=0}^n a^j,\sum_{j=0}^n a^j\rangle = 0$ gives

$$b_{i-1}^n\cdot c_{n,i}^2 - 2c_{n,i}\left(b_{i-2}^{n-1} + b_{i-1}^{n-1}\cdot\|a^0\|\right) + b_{i-2}^{n-2}\cdot 2\|a^0\| + b_{i-3}^{n-2} = 0$$

Since the right hand side of (4.2) is equal to 0 we get $\|a^0\| = 1/(n+1)$.

Now suppose $\|\sum_{j\in M_i} a^j\| < d_{n,i}$ for all $M_i\in\mathcal{M}_i^n$. Taking the square and summing over all $M_i\in\mathcal{M}_i^n$ yields

$$\begin{aligned}
b_{i-1}^n\cdot\sum\nolimits_{j=0}^n\|a^j\|^2 + b_{i-2}^{n-1}\cdot 2\sum\nolimits_{0\le k<j\le n}\langle a^k,a^j\rangle < b_i^{n+1}\cdot d_{n,i}^2 \iff\\
b_{i-1}^{n-1}\cdot\sum\nolimits_{j=0}^n\|a^j\|^2 + b_{i-2}^{n-1}\cdot\|\sum\nolimits_{j=0}^n a^j\|^2 < b_i^{n+1}\cdot d_{n,i}^2.
\end{aligned}$$

Dividing by b_i^{n+1} gives

$$\frac{i(n+1-i)}{n(n+1)}\sum_{j=0}^n\|a^j\|^2 + \frac{i(i-1)}{n(n+1)}\|\sum_{j=0}^n a^j\|^2 < d_{n,i}^2. \tag{4.3}$$

By $\sum_{j=0}^n\|a^j\| = 1$ we have $\sum_{j=0}^n\|a^j\|^2 \ge 1/(n+1)$ and this shows ii). If we have equality in this inequality we must have $\|\sum_{j\in M_i} a^j\| = d_{n,i}$ for all $M_i\in\mathcal{M}_i^n$. By (4.3) we see $\sum_{j=0}^n a^j = 0$ and $\sum_{j=0}^n\|a^j\|^2 = 1/(n+1)$. This show $\|a^k\| = \|a^j\|$, $0\le k\le n$, and thus $\|a^k\| = 1/(n+1)$. □

By this theorem we can easily deduce lower bounds for the ratio $r(T)/r_i(T)$. We will divide our results in two Corollaries, because for the cases $i = d-1$ or $i = 1$ we have best possible bounds, but not for the others.

Corollary 4.1. *Let $T \subset E^d$ be a d-dimensional simplex. Then*

$$r(T)/r_{d-1}(T) \geq \beta(d, d-1),$$
$$r(T)/r_1(T) \geq \beta(d, 1),$$

and equality holds if and only if T is regular.

Proof. We may assume $\sum_{j=0}^{d} \|u^j\| = 1$. If $C = (C_0, \ldots, C_{d-1}) \in \mathcal{C}_{d-1}^d$ is a $(d-1)$-partition then we must have one subset (say C_0) with two elements and all other subsets C_j consist of one element. By the identity above we have $\sum_{j=1}^{d} \sum_{k \in C_j} \|u^k\| = 1 - \sum_{k \in C_0} \|u^k\|$ and thus by Theorem 3.1. and Theorem 4.1. i)

$$r(T)/r_{d-1}(T) = 1 - \min_{M_2 \in \mathcal{M}_2^d} \left(\sum_{j \in M_2} \|u^j\| - \|\sum_{j \in M_2} u^j\| \right) \geq \beta(d, d-1). \tag{4.4}$$

Now, if $C = (C_0, C_1) \in \mathcal{C}_1^d$ is a 1-partition we have by (1.2) $\|\sum_{j \in C_1} u^j\| = \|\sum_{j \in C_0} u^j\|$. On account of Theorem 3.1. and Theorem 4.1. ii) we get

$$\begin{aligned} r(T)/r_1(T) &= \max_{(C_0, C_1) \in \mathcal{C}_1^d} 2\|\sum_{j \in C_0} u^j\| = \max_{1 \leq k \leq d} \max_{M_k \in \mathcal{M}_k^d} 2\|\sum_{j \in M_k} u^j\| \\ &\geq 2 \max_{1 \leq k \leq d} \sqrt{k(d+1-k)/((d+1)^2 d)}. \end{aligned} \tag{4.5}$$

The right hand side is maximal for $k = d/2$ if d is even, for $k = (d+1)/2$ else.

If we have equality in one of the inequalities we must have equality in (4.4) or (4.5). Hence we have equality in the appropriate relations of Theorem 4.1. Thus $\|u^j\| = 1/(n+1)$, $0 \leq j \leq d$, and this is only possible if T is regular. By Lemma 3.2. we see that we have equality for a regular simplex. □

If we consider for $1 < i < d-1$ the i-partitions with the property that one subset has $(d+1-i)$ elements and the other sets consist of exactly one element we can apply Theorem 4.1. in the same way as in the first part of Corollary 4.1. and get

Corollary 4.2. *Let $T \subset E^d$ be a d-dimensional simplex. Then for $1 < i < d-1$*

$$r(T)/r_i(T) \geq \alpha(d, i).$$

It seems to be quite likely that these lower bounds are not best possible and Corollary 4.1. suggests that $r(T)/r_i(T)$ becomes maximal for simplices if T is regular. Hence we believe (cf. Lemma 3.2.)

Conjecture 4.1. *Let $T \subset E^d$ be a d-dimensional simplex. Then for $1 \leq i \leq d$*

$$r(T)/r_i(T) \geq \beta(d, i),$$

and equality holds if and only if T is regular.

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5. Minimal inradii of convex bodies

By the results of the last section we can easily obtain estimates of $r_j(K)/r_i(K)$ for $K \in \mathcal{K}^d, d \geq j \geq i \geq 1$. For convenience we write

$$r_{ji}^d = \inf\{r_j(K)/r_i(K) \mid K \in \mathcal{K}^d,\, r_1(K) < \infty\}.$$

First we observe that it is sufficient to study r_{di}^d:

Lemma 5.1. *Let $d \geq j \geq i \geq 1$. Then $r_{ji}^d = r_{ji}^j$.*

Proof. For every $\epsilon > 0$ there is by definition a $K \in \mathcal{K}^d$, such that $r_j(K)/r_i(K) \leq r_{ji}^d + \epsilon$. For this K there is a $L \in \mathcal{L}_j^d$ such that $r_j(K) = r(K|L; L)$. Let L' denote the orthogonal complement of L and let $K' = (K|L) + L'$. Then $K \subset K'$ and thus $r_k(K') \geq r_k(K)$ for $1 \leq k \leq d$ and evidently $r_j(K') = r_j(K)$. Now let $M \in \mathcal{L}_i^d$. We set $M' = M \cap L, M''$ the orthogonal complement of M' with respect to M. We have $K'|M = (K'|M') + M''$. This shows that there is a $M \in \mathcal{L}_i^d$ with $r_i(K') = r(K'|M; M)$ and $M \subset L$. Altogether we have

$$r_{ji}^d + \epsilon \geq r_j(K')/r_i(K') = r_j(K' \cap L; L)/r_i(K' \cap L; L) \geq r_{ji}^j.$$

We can easily reverse the argument: For a $K \in \mathcal{K}^j$ with $r_j(K)/r_i(K) \leq r_{ji}^j + \epsilon$ we regard E^j as an element $L \in \mathcal{L}_j^d$ with complement L'. Then $K + L'$ shows in the same way as before $r_{ji}^j + \epsilon \geq r_{ji}^d$. $\square$

Now we can finally prove our main theorem

Theorem 5.1. *Let $K \in \mathcal{K}^d$ be a d-dimensional convex set with $r_1(K) < \infty$. Then for $2 \leq j \leq d$*

$$r_j(K)/r_i(K) \geq \beta(j,i), \quad i = 1 \text{ or } i = j-1,$$
$$r_j(K)/r_i(K) \geq \alpha(j,i), \quad 1 < i < j-1,$$

and for $i \in \{1, j-1\}$ these inequalities are best possible.

Proof. On account of Lemma 5.1. it is sufficient to treat the case $j = d$. Let 0 be the center of the inscribed ball of K with radius $r(K)$. Then there exists a m-dimensional simplex $T \subset K$, such that $0 \in \operatorname{relint}(T)$ and the vertices of T belong to the boundary of K [BF].

Let $L_m = \operatorname{lin}(T)$. The support planes on the inscribed ball with respect to the vertices of T form a cylinder P with $K \subset P$ and $T = P \cap L_m$ is a m-dimensional simplex with $r(K) = r(T; L_m)$. Further we have $K|L_m \subset T$ and hence

$$r(K) = r_i(K), \quad m \leq i \leq d.$$

Now the right hand sides in the Theorem are less 1 and hence we only have to consider the cases $1 \leq i \leq m-1$. We have $r_i(T; L_m) \geq r_i(K|L_m; L_m) \geq r_i(K)$ and thus $r(K)/r_i(K) \geq r(T; L_m)/r_i(T; L_m)$ for $1 \leq i \leq m-1$. On account of the Corollaries 4.1. and 4.2. we get

$$r(K)/r_i(K) \geq \beta(m,i) \geq \beta(d,i), \quad i = 1 \text{ or } i = m-1,$$
$$r(K)/r_i(K) \geq \alpha(m,i) \geq \alpha(d,i), \quad 1 < i < m-1,$$

where the inequalities $\beta(m,i) \geq \beta(d,i)$ and $\alpha(m,i) \geq \alpha(d,i)$ are checked by elementary computation. By Corollary 4.1. we see that the inequalities $i = 1 \vee i = j-1$ are best possible. $\square$

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